Nonequilibrium dynamics of language games on complex networks

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The naming game is a model of nonequilibrium dynamics for the self-organized emergence of a linguistic convention or a communication system in a population of agents with pairwise local interactions. We present an extensive study of its dynamics on complex networks, that can be considered as the most natural topological embedding for agents involved in language games and opinion dynamics. Except for some community structured networks on which metastable phases can be observed, agents playing the naming game always manage to reach a global consensus. This convergence is obtained after a time generically scaling with the population's size N as $t_{conv} \sim N^{1.4 \pm 0.1}$, i.e., much faster than for agents embedded on regular lattices. Moreover, the memory capacity required by the system scales only linearly with its size. Particular attention is given to heterogenous networks, in which the dynamical activity pattern of a node depends on its degree. High-degree nodes have a fundamental role, but require larger memory capacity. They govern the dynamics acting as spreaders of (linguistic) conventions. The effects of other properties, such as the average degree and the clustering, are also discussed.

I. INTRODUCTION

Understanding the origin and the evolution of language or more generally of communication systems is a fascinating challenge for the interdisciplinary scientific community [1], demanding contributions to researchers in very different fields, from linguistics to artificial intelligence, from social sciences to biology, mathematics and physics. Even though a unitarian view of language as a complex system is still lacking, a number of different approaches has been proposed in order to get major insights into some specific aspects such as, for instance, the self-organized processes leading to the emergence of a shared lexicon (i.e., a communication system) in a population of agents. In the past few years, it has been shown that simple models of interacting agents could display a collective agreement on a shared mapping between words and objects (or meanings), eventually bootstrapping a shared system of linguistic conventions, even without global supervision or *a priori* common knowledge [2–4]. Such models of "language games," in which the organization of language is tackled at a purely semiotic level (neglecting semantic relations between symbols and meanings), have played a pivotal role for the understanding of emergent communication systems. We will restrain ourselves to the case of the emergence and evolution of a communication system on short temporal scales compared to those involved in the evolution of language, so that we neglect any darwinian principle commonly used in the modelization of language evolution [5,6], and focus on simple population dynamics. In this context, a new field of research called semiotic dynamics has been recently developed, that investigates by means of simple models how (linguistic) conventions originate, spread and evolve over time in a population of agents endowed with basic internal states and able to perform local pairwise interactions [7].

The fundamental model of semiotic dynamics is the naming game [8], that describes a population of agents trying to agree on the assignation of names to objects. The emergence of consensus about the object's name allows to establish a communication system. Such a model was inspired by global coordination problems in artificial intelligence and by peer-to-peer communication modeling. A practical example of this type of dynamics is provided by the Talking Heads experiment [9], in which embodied software agents develop their vocabulary observing objects through digital cameras, assigning them randomly chosen names and sharing these names in pairwise interactions. Very recently [10,11], models of semiotic dynamics have as well found an application in the study of a new generation of web-tools which enable human web-users to self-organize a system of tags in such a way to ensure a shared classification of information about different arguments (see, for instance, del.icio.us or www.flickr.com).

Statistical physics has been involved in the analysis of models of emergent collective behavior in interacting particles systems for a long time. It is, therefore, not a surprise that, recently, various contributions have come from physicists, in order to shed light on the dynamics of opinion formation through the study of models of social interactions [12]. The naming game (NG), as a model of interacting agents reaching a global consensus through emergent cooperative phenomena, can as well be studied through the statistical physics toolbox that can give insights into the corresponding complex dynamics. As a first natural step, previous studies have considered, as was the case in the Talking Heads experiments, that each agent was allowed to interact with all the others [13]. This mean-field-like scenario can indeed be realistic when dealing with a small number of agents. Moreover, the case of agents embedded into low-dimensional lattices has as well been investigated [14], showing that the global behavior of the naming game strongly depends on the underlying topology. Recently, however, the growing field of complex networks [15–17] has allowed to obtain a better knowledge of social networks [18], and, in particular, to show that the typical topology of the networks on which
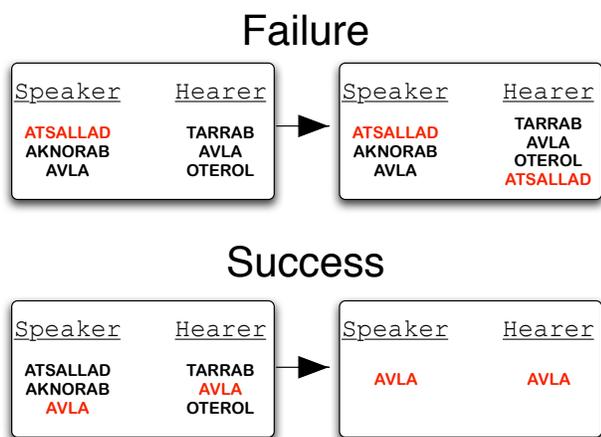
FIG. 1. (Color online) Agents' interaction rules. Each agent is described by its inventory, i.e., the repertoire of known words. The speaker picks up at random a name in its inventory and transmits it to the hearer. If the hearer does not know the selected word the interaction is a failure (top), and it adds the new name to its inventory. Otherwise (bottom) the interaction is a success and both agents delete all their words but the winning one. Note that if the speaker has an empty inventory (as it happens at the beginning of the game), it invents a new name and the interaction is, of course, a failure.

agents interact is not regular. The natural step taken in this paper is thus to consider the naming game for agents embedded on more realistic networks and to study the influence of various complex topologies on the corresponding dynamical behavior.

The paper is organized as follows. Sections II and III are devoted to the definition of the model and to summarize already known results about the naming game dynamics in the case of mean-field and low-dimensional lattices. In Section IV, we briefly recall the definition of network models subsequently used in the theoretical and numerical analysis, whose results are exposed in Section V. Conclusions and directions for future works are exposed in Section. VI.

II. MODEL DEFINITION

A minimal model of naming game has been put forward by Baronchelli *et al.* in Ref. [13] to reproduce the main features of semiotic dynamics and the fundamental results of adaptive coordination observed in the talking heads experiment [9]. In this minimal model, N identical agents observe a single object, for which they invent names that they try to communicate to one another through pairwise interactions. Each agent is endowed with an internal inventory, in which it can store an *a priori* unlimited number of names (or opinions). All agents start with empty inventories. At each time step, a pair of neighboring agents is chosen randomly, one playing as "speaker" the other as "hearer," and a negotiation process takes place according to the following rules (see also Fig. 1). The speaker transmits a name to the hearer. If its inventory is empty, a new word is invented, otherwise it selects randomly one of the names it knows. If the hearer has

the uttered name in its inventory, the game is a success, and both agents delete all their words but the winning one. If the hearer does not know the uttered word, the game is a failure, and the hearer adds the word to its inventory, i.e., it learns it.

Note that the time unit is here given by one interaction, in contrast to most nonequilibrium statistical physics models in which a time unit corresponds to N interactions. In many cases, results for the dynamical evolution will therefore be expressed as a function of the rescaled time t/N.

Although this model can be seen as belonging to the broad class of opinion formation models, it is interesting to note the important differences with other commonly studied such models [12]. In particular, each agent can potentially be in an infinite number of possible discrete states (or words, names), contrarily to the Voter model in which each agent has only two possible states. Moreover, an agent can here accumulate in its memory different possible names for the object, i.e., wait before reaching a decision, and has an *a priori* unlimited memory. An interesting question therefore relates to the actual memory size required by each agent during the dynamics. Finally, each dynamical step can be seen as a negotiation between speaker and hearer, with a certain degree of stochasticity, while in the Voter model, an agent deterministically adopts the opinion of one of its neighbors.

Another remark concerns the random extraction of the word to be uttered from the speaker's inventory. Previously proposed models of semiotic dynamics used a more complicated representation of the negotiation interaction assigning weights to the words in the inventories. In such models (see [19], and references therein), the word with largest weight is automatically chosen by the speaker and communicated to the hearer. Communicative success or failures are translated into updates of the weights: the weight of a word involved in a successful interaction is increased to the detriment of the other weights (with no deletion process), while a failure leads to the decrease of the weight of the word not understood by the hearer. While such rules are certainly more realistic than the drastic deletion rule of the minimal naming game, the latter has been shown to retain the essential features of the emergence of a global collective behavior and corresponds to a much simpler definition.

It is also worth noting that in the minimal naming game all agents refer to the same single object, while in the original experiments the embodied agents could observe a set of different objects. This is due to the assumption that homonymy is excluded, i.e., it is impossible that two distinct objects assume the same name. Thus, in the model, all objects are independent and the general problem reduces to a set of uncorrelated systems, each one described by the minimal model.

In the rest of the paper all analysis and numerical simulations will deal with this simplified model that can be rightly seen as the prototype of the naming game.

III. STATE OF THE ART

Most previous studies in semiotic dynamics have focused on populations of agents in which all pairwise interactions
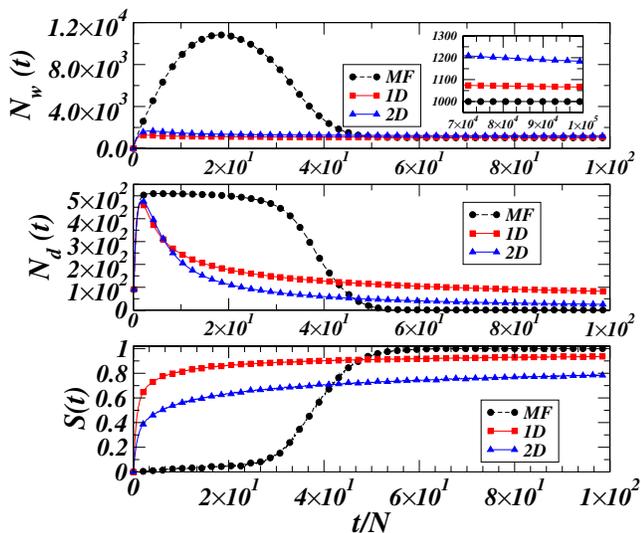
FIG. 2. (Color online) Evolution of the total number of words N_w (top), of the number of different words N_d (middle), and of the average success rate $S(t)$ (bottom), for a mean-field system (black circles) and low-dimensional lattices (1D, red squares and 2D, blue triangles) with $N=1024$ agents, averaged over 10^3 realizations. The inset in the top graph shows the very slow convergence for low-dimensional systems.

are allowed, i.e., the agents are placed on the vertices of a fully connected graph. In statistical mechanics, this topological structure is commonly referred to as "mean-field" topology. In the original work on the minimal naming game model [13], Baronchelli *et al.* have studied numerically and analytically the behavior of the mean-field model, providing theoretical arguments in order to explain the main properties of the population's global behavior. The overall dynamics has been studied monitoring the temporal evolution of the total number $N_w(t)$ of words in the system at time t, i.e., the total memory used by the agents' inventories, of the number of different words $N_d(t)$, and of the average success rate $S(t)$ (i.e., the probability, computed averaging on many simulation runs, that the interaction at time t is successful). At the beginning, many disjoint pairs of agents interact, with empty initial inventories: they invent a large number of different words that then start spreading throughout the system, through unsuccessful interactions. Indeed, in the early stages of the dynamics, the overlap between inventories is very low and successful interactions are limited to those pairs which have been chosen at least twice. Since the number of possible partners of an agent is of order N, an agent rarely interacts twice with the same partner: the probability of such an event grows only as t/N^2. The consequence, shown in Fig. 2 (black circles), is that in this phase the number of different words N_d invented by the agents grows, reaching a maximum that scales as $O(N)$. N_d then saturates (no inventory is empty anymore so that no new words are invented) and displays a plateau, while the total number N_w of words keep growing since the various words propagate in the system and correlations grow between inventories. The peak of N_w has been shown to scale as $O(N^{1.5})$ [13], which means that each agent stores $O(N^{0.5})$ words, and occurs after the

system has evolved for a time $t_{max} \sim O(N^{1.5})$. The strong correlations built during this time finally lead the system to consensus in a time t_{conv} of order $N^{1.5}$. The final state corresponds to the agreement of the agents on the name to be assigned to a particular object: $N_w = N$, which means that each agent possesses a unique word in its inventory, and $N_d = 1$, which shows that this word is the same for all agents. The S-shaped curve (black circles) of the success rate in Fig. 2 summarizes the dynamics: initially, agents hardly understand each other ($S(t)$ is very low); the inventories start to present significant overlaps, so that $S(t)$ increases until it reaches 1, and the communication system is completely set in.

A first study of the effects of topological embedding on the naming game dynamics is reported in Ref. [14]. When the interacting agents sit on the nodes of low-dimensional lattices, the long-time behavior is still characterized by the convergence to a homogeneous consensus state, but the evolution of the system changes considerably. Since each agent can interact only with a limited number of neighbors ($2d$ in a d-dimensional lattice), at the local scale the dynamics is very fast: agents can rapidly interact two or more times with their neighbors, favoring the establishment of a local consensus with a high success rate (Fig. 2, red squares for one-dimensional (1D) and blue triangles for 2D), i.e., of small sets of neighboring agents sharing a common unique word. These "clusters" of neighboring agents with a common unique word are separated by individuals having a larger inventory with two or more words, playing the role of "interfaces." For one-dimensional systems, it can be proved analytically [14] that the motion of interfaces is a random walk for which the diffusion coefficient can be computed. Consequently, the clusters of unique words grow in time with a law that is typical of coarsening phenomena [20], i.e., the competition among the clusters is driven by the fluctuations of the interfaces. The coarsening picture can be extended to higher dimensions, where it has been checked numerically. Such an analysis shows that in low-dimensional structures local consensus is easy but in the long run delays the global consensus, which takes much longer to be reached than in mean field (see Fig. 2): for example, $O(N^3)$ in dimension 1 vs $O(N^{1.5})$ in mean field. However, another important aspect of the problem concerns the memory used by the agents. In mean field indeed, each agent needs a memory capacity scaling as $O(N^{1/2})$, i.e., diverging with the system size. In contrast, the consequence of the embedding in a finite-dimensional lattice (with a finite number of neighbors), and of the subsequent coarseninglike phenomena, with rapid local consensus, is that each agent uses only a finite capacity: the maximum total number of words in the system (maximal memory capacity) scales linearly with the system size N (as for the number of different words). In summary, *low-dimensional lattice systems require more time to reach the consensus compared to mean field, but a lower use of memory.*

Social interactions, however, take place on networks that are neither mean-field-like nor regular lattices, but share a certain number of properties such as the small-world property (the average topological distance between nodes in-

creases very slowly—logarithmically or even slower—with the number of nodes) and the relative abundance of "hubs" (nodes with very large degrees compared to the means of the degree distribution $P(k)$). More precisely, the degree distributions are in many cases heterogeneous, with heavy tails often of a power-law (or "scale-free") nature (for a significant range of values of k, one has $P(k) \sim k^{-\gamma}$ [15–17]. Moreover, social networks are often characterized by a large transitivity, which implies that two neighbors of a given vertex are also connected to each other with large probability. Transitivity can be quantitatively measured by means of the clustering coefficient c_i of vertex i [21], defined as the ratio between the number of edges m_i existing between the k_i neighbors of i, and its maximum possible value, i.e., $c_i = 2m_i/[k_i(k_i-1)]$. The average clustering coefficient, defined as $C = \Sigma_i c_i/N$, usually takes quite large values in real complex networks.

In order to investigate to what extent these properties can affect the local and global dynamics of the naming ame, we have performed extensive simulations of this model with agents embedded on the nodes of various paradigmatic computer-generated network models, whose definitions and main properties are recalled in the next section.

IV. NETWORKS DEFINITION

While many models with various characteristics have been proposed in the last years in order to account for various detailed properties of real-world networks, our aim in this paper is to understand the influence on the dynamics of the naming game of the most salient properties such as heterogeneity in the degree distribution, clustering, average degree, and we will therefore concentrate on a few network models that have become indeed paradigms of complex networks.

The prototype of homogeneous networks is the uncorrelated random graph model proposed by Erdös and Rényi (ER model) [22], whose construction consists in drawing an (undirected) edge with a fixed probability p between each possible pair out of N given vertices. The resulting graph shows a binomial degree distribution with average $\langle k \rangle \simeq Np$, converging to a poissonian distribution for large N. If p is sufficiently small (order $1/N$), the graph is sparse and presents locally tree-like structures.

In order to account for degree heterogeneity, other constructions have been proposed for random graphs with arbitrary degree distributions [23–26]. In particular, we will consider the uncorrelated configuration (UC) model which yields uncorrelated random graphs through the following construction: N vertices with a fixed degree sequence taken from the desired degree distribution, with a cut-off $\sqrt{N}$, are connected randomly avoiding multi-links and self-links.

Since many real networks are not static but evolving, with new nodes entering and establishing connections to already existing nodes, many models of growing networks have also been introduced. We will consider the model introduced by Barabási and Albert (BA) [27], which has become one of the most famous models for complex heterogeneous networks, and is constructed as follows: starting from a small set of m interconnected nodes, new nodes are introduced one by one. Each new node selects m older nodes according to the *preferential attachment* rule, i.e., with probability proportional to their degree, and creates links with them. The procedure stops when the required network size N is reached. The obtained network has an average degree $\langle k \rangle = 2m$, small clustering (of order $1/N$) and a power-law degree distribution $P(k) \sim k^{-\gamma}$, with $\gamma = 3$.

The BA networks have small clustering, in contrast with social networks. It turns out that growing networks can as well be constructed with a large clustering. In Ref. [28] indeed, Dorogovtsev *et al.* have proposed a model (DMS model) in which each new node connects with the two extremities of a randomly chosen edge, forming therefore a triangle. Since the number of edges arriving to any node is, in fact, its degree, the probability of attaching the new node to an old node is proportional to its degree and the preferential attachment is recovered. The degree distribution is, therefore, the same as the one of a BA model with $m=2$, and the degree-degree correlations are as well equal. However, the clustering coefficient is large and approximately equal to 0.73 [29]. In order to tune the clustering, we can consider a generalization of this construction, in the spirit of the Holme-Kim model [30]: starting from m connected nodes (with m even), a new node is added at each time step; with probability q it is connected to m nodes chosen with the preferential attachment rule (BA step), and with probability $1-q$ it is connected to the extremities of $m/2$ edges chosen at random (DMS-like step). The one-node and two-node properties (i.e., degree distribution and degree-degree correlations) are the same as the ones of the BA network, while the clustering spectrum, i.e., the average clustering coefficient of nodes of degree k, can be computed as $C(k) = 2(1-q)(k-m)/[k(k-1)] + O(1/N)$ [29,31]: changing m and q allows to tune the value of the clustering coefficient.

Since the ER model also displays a low clustering, we consider, moreover, a purposely modified version of this random graph model (clustered ER, or CER model) with tunable clustering. Given N nodes, each pair of nodes is considered with probability p; the two nodes are then linked with probability $1-Q$ while, with probability Q, a third node (which is not already linked with either) is chosen and a triangle is formed. The clustering is thus proportional to Q (with $p \sim O(1/N)$ we can neglect the original clustering of the ER network) while the average degree is approximately given by $\langle k \rangle \simeq [3Q + (1-Q)]pN \simeq (2Q+1)pN$ [43].

The next section contains the results of simulations of the minimal naming game with agents embedded on ER and BA networks. Our simulations have been carried out on networks of sizes ranging from 10^3 to 5×10^4 nodes, with results averaged over 20 runs per network realization and over 20 network realizations. Since the BA model has some particular hierarchical structure due to its growing construction, we have compared the corresponding results with the case of networks created with the UC model, in which the exponent of the degree distribution can moreover be varied. It turns out that the obtained behavior is very similar, so that we will display results for the BA model. The effect of clustering will be discussed using the mixed BA-DMS and the CER network models.

V. RESULTS

In this section we expose the main results on the dynamics of the naming game on complex networks. Before entering into the details of the analysis, it is worth noting that the minimal naming game model itself, as described in Sec. II, is not well-defined on general networks. Indeed, the two neighboring agents chosen to interact have different roles: one (the speaker) transmits a word and is thus more "active" than the other (the hearer). One should, therefore, specify whether, when choosing a pair, one chooses first a speaker and then a hearer among the speaker's neighbors, or the reverse order. If the agents sit on either a fully connected graph or on a regular lattice, they have an equivalent neighborhood so the order is not important. On a generic network with degree distribution $P(k)$, however, the degree of the first chosen node and of its chosen neighbor is distributed, respectively, according to $P(k)$ and to $kP(k)/\langle k \rangle$. The second node will, therefore, have typically a larger degree, and the asymmetry between speaker and hearer can couple to the asymmetry between a randomly chosen node and its randomly chosen neighbor, leading to different dynamical properties (this is the case, for example, in the Voter model, as studied by Castellano [32]). This is particularly relevant in heterogeneous networks for which a neighbor of a randomly chosen node is a hub with relatively large probability. We, therefore, can distinguish more possibilities for the definition of the naming game on generic networks.

(i) A randomly chosen speaker selects (again randomly) a hearer among its neighbors. This is probably the most natural generalization of the original rule. We call this strategy *direct naming game*. In this case, larger degree nodes will preferentially act as hearers.

(ii) The opposite strategy, here called *reverse naming game*, can also be carried out: we choose the hearer at random and one of its neighbors as speaker. In this case the hubs are preferentially selected as speakers.

(iii) A *neutral* strategy to pick up pairs of nodes is that of considering the extremities of an edge taken uniformly at random. The role of speaker and hearer are then assigned randomly with equal probability among the two nodes.

Figure 3 allows us to compare the evolution of the direct and the reverse naming game for a BA network of $N=10^4$ agents and $\langle k \rangle = 4$. In the case of the reverse rule, a larger memory is used although the number of different words created is smaller, and a faster convergence is obtained. This corresponds to the fact that the hubs, playing principally as speakers, can spread their words to a larger fraction of the agents, and remain more stable than when playing as hearers, enhancing the possibility of convergence. Depending on the network under study, and similarly to the Voter model case [32], the scaling laws of the convergence time can even be modified, as our preliminary study shows. A detailed analysis of this behavior remains, however, beyond the scope of our present study and we leave it for future work (see also [33]). From the point of view of a realistic interaction among individuals or computer-based agents, the direct naming game in which the speaker chooses a hearer among its neighbors seems somehow more natural than the other ones. In the
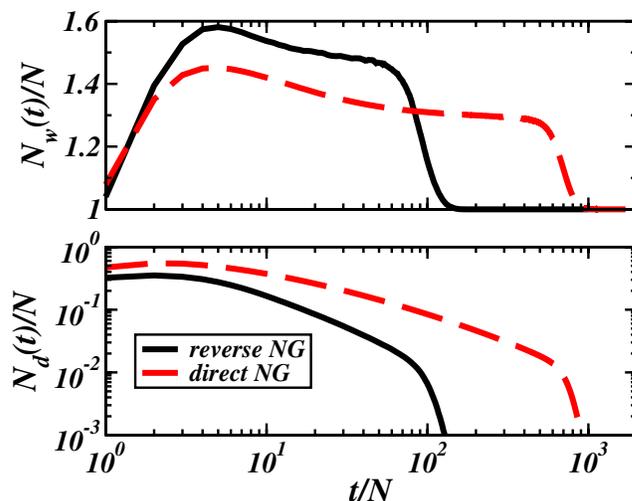
FIG. 3. (Color online) Total memory N_w (top) and number of different words N_d (bottom) vs rescaled time for two different strategies of pair selection on a BA network of $N=10^4$ agents, with $\langle k \rangle = 4$. The reverse NG rule (black full line) converges much faster than the direct rule (red dashed line). Nonetheless, the two strategies lead to the same scaling laws with the system size for the convergence time (not shown).

remainder of this paper therefore, we will focus on the direct naming game, mentioning, where necessary, the corresponding behavior for the other two rules.

A. Global quantities

We first study the global behavior of the system through the temporal evolution of three main quantities: the total number $N_w(t)$ of words in the system, the number of different words $N_d(t)$, and the rate of success $S(t)$. All these quantities are averaged over a large number of runs and networks realizations. In Fig. 4, we report the curves of $N_w(t)$ and $N_d(t)$ for ER (left) and BA networks (right) with $N=10^3$, 10^4, and 5×10^4 nodes and average degree $\langle k \rangle = 4$. The corresponding data for the mean-field case (with $N=10^3$) are displayed as well for reference. The curves for the average use of memory $N_w(t)$ show a rapid growth at short times, a peak, and then a plateau whose length increases as the size of the system is increased (even when the time is rescaled by the system size, as in Fig. 4). The time and height of the peak, and the height of the plateau, are proportional to N. These scaling properties are systematically studied in Fig. 5, which also shows that the convergence time t_{conv} scales as $N^{1.4}$ for both ER and BA. The apparent plateau of N_w does, however, not correspond to a steady state, as revealed by the continuous decrease of the number of different words N_d in the system: in this reorganization phase, the system keeps evolving by elimination of words, although the total used memory almost does not change.

The scaling laws observed for the convergence time is a general robust feature that is not affected by further topological details, such as the average degree, the clustering, or the particular form of the degree distribution. We have checked the value of the exponent 1.4 ± 0.1 for various $\langle k \rangle$, clustering,
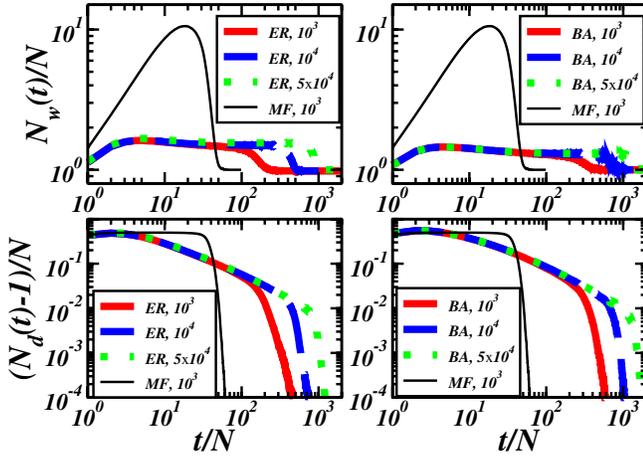
FIG. 4. (Color online) ER random graph (left) and BA scale-free network (right) with $\langle k \rangle = 4$ and sizes $N = 10^3, 10^4, 5 \times 10^4$. Top: evolution of the average memory per agent N_w/N versus rescaled time t/N. For increasing sizes a plateau develops in the reorganization phase preceding the convergence. The height of the peak and of the plateau collapse in this plot, showing that the total memory used scales with N. Bottom: evolution of the number of different words N_d in the system. $(N_d - 1)/N$ is plotted in order to emphasize the convergence to the consensus with $N_d = 1$. A steady decrease is observed even if the memory N_w displays a plateau. The mean-field (MF) case is also shown (for $N = 10^3$) for comparison.

and exponents γ of the degree distribution $P(k) \sim k^{-\gamma}$ for scale-free networks constructed with the uncorrelated configuration model. All these parameters have instead an effect on the other quantities such as the time and the value of the maximum of memory (see Sec. V D).

Figures 2, 4 and 5 allow for a direct comparison between the networks investigated and both the mean-field (MF) to-
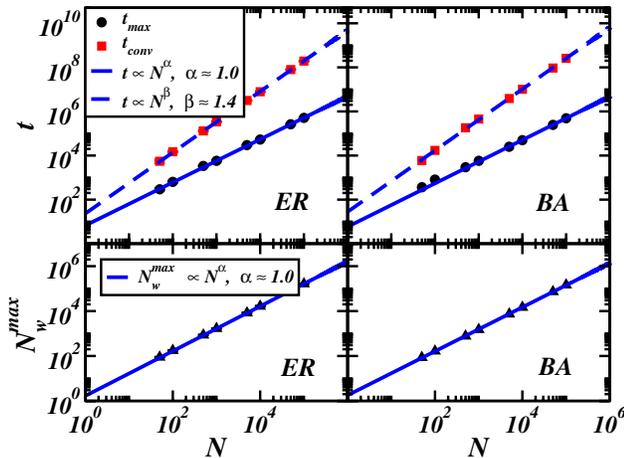
FIG. 5. (Color online) Top: scaling behavior with the system size N for the time of the memory peak (t_{max}) and the convergence time (t_{conv}) for ER random graphs (left) and BA scale-free networks (right) with average degree $\langle k \rangle = 4$. In both cases, the maximal memory is needed after a time proportional to the system size, while the time needed for convergence grows as N^β with $\beta \simeq 1.4$. Bottom: In both networks the necessary memory capacity (i.e., the maximal value N_w^{max} reached by N_w) scales linearly with the size of the network.
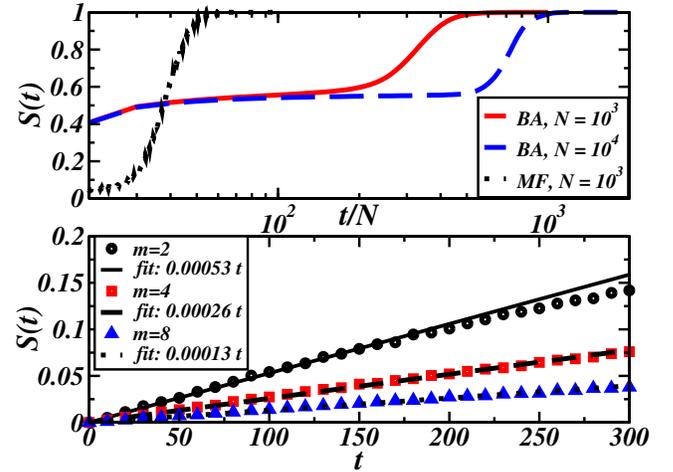
FIG. 6. (Color online) Top: Temporal evolution of the success rate $S(t)$ for BA scale-free networks with $\langle k \rangle = 4$ and sizes $N = 10^3$ and 10^4. The dotted black line refers to the mean-field case ($N = 10^3$). The bottom plot shows the short time behavior for $N = 10^3$ and $\langle k \rangle = 4$ ($m = 2$, circles), $\langle k \rangle = 8$ ($m = 4$, squares), and $\langle k \rangle = 16$ ($m = 8$, triangles). The fitted slopes are very close to the predicted value $2/\langle k \rangle N$.

pology and the regular lattices. Thanks to the finite average connectivity, the memory peak scales only linearly with the system size N, and is reached after a time $O(N)$, in contrast with MF ($O(N^{1.5})$ for peak height and time) but similarly to the finite-dimensional case. The MF plateau observed in the number of different words, and corresponding to the building of correlations between inventories, with an increasing used global memory, and almost no cancellation of words, is replaced here by a slow continuous decrease of N_d with an almost constant memory used. With respect to the slow coarsening process observed in finite-dimensional lattices, on the other hand, the small-world properties of the networks, i.e., the existence of short paths among the nodes, speeds up the convergence towards the global consensus (see also [34]). Therefore, complex networks exhibiting small-world properties constitute an interesting trade-off between mean-field "temporal efficiency" and regular lattice "storage optimization."

Figure 6 displays the success rate $S(t)$ for BA networks with $N = 10^3$ (red full line), and 10^4 (blue dashed line) agents and $\langle k \rangle = 4$. The behavior for ER networks is similar. The success rate for the mean-field ($N = 10^3$) is also reported (black dotted lines). The success rate increases linearly at very short times (bottom plot of Fig. 6) then, after a plateau similar to the one observed for N_w, increases on a fast time scale towards 1. At short times most inventories are empty, so that the success rate is equal to the probability that two agents interact twice, i.e., t/E, where $E = N\langle k \rangle/2$ is the number of possible interacting pairs (i.e., the number of links in the network), as shown in Fig. 6 for BA networks where linear fits to $S(t)$ give slopes in agreement with the theoretical prediction $2/\langle k \rangle N$. Note that this argument as well explains that in mean-field the initial success rate is much lower than for finite $\langle k \rangle$, since there $E = N(N-1)/2 = O(N^2)$. When $t \sim O(N)$, no inventory is empty anymore, words start
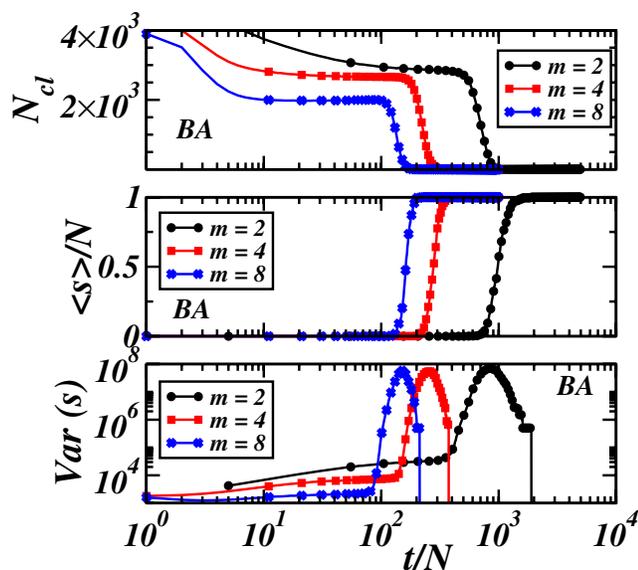
FIG. 7. (Color online) BA network with $N=10^4$, $\langle k \rangle=4$ (circles), $\langle k \rangle=8$ (squares), $\langle k \rangle=16$ (crosses). From top to bottom: Total number of clusters, average normalized cluster size $\langle s \rangle/N$, fluctuations of the cluster size vs time.

FIG. 8. (Color online) ER and BA networks with average degree $\langle k \rangle=4$. Fractions of nodes with one (n_1 in black), two (n_2 in red), and three (n_3 in blue) words versus time. These fractions evolve only very slowly during the plateau displayed by the memory. The curves for three different sizes are reported: $N=10^3$ (dotted line), $N=10^4$ (dashed line), and $N=5 \times 10^4$ (full line).

spreading through unsuccessful interactions and $S(t)$ displays a bending.

B. Clusters statistics

We now turn our attention to a complementary aspect of the dynamics of the naming game: the behavior of clusters of words. We call "cluster" any set of neighboring agents sharing a common unique word. In the case of agents embedded in low-dimensional lattices, it has indeed been shown [14] that the dynamics of the naming game proceeds by formation of such clusters, that grow through a coarsening phenomenon: the average cluster size (respectively, the number of clusters) increases (respectively, decreases) algebraically with time. On generic networks, a different behavior can be expected. As shown indeed in Fig. 7 for the BA model (the behavior is very similar for the ER model) the number of clusters reaches very rapidly a plateau that lasts up to the convergence time at which it suddenly falls to 1. Moreover, the normalized average cluster size remains very close to zero (in fact, of order $1/N$) during the plateau, and converges to one with a similar sudden transition. This transition becomes steeper when the average degree increases (and also when the size of the system increases), as also emphasized by sharper peaks in the variance of the cluster size.

In the same spirit, it is interesting to monitor the number of agents with a certain number of words: agents with only one word are parts of clusters while agents using more memory are propagating words from one part of the system to another. Figure 8 shows the temporal evolution of the fractions of nodes with 1, 2, and 3 words. As for N_w and the cluster size, these quantities display plateaus whose length increases with the system size (even in rescaled time units t/N), and converge, respectively, to 1 and 0 abruptly at t_{conv}. Moreover, n_1 is much lower than what would be observed in

a coarsening process in which agents with more than one word are only found at the interfaces.

The emerging picture is very different from the coarsening obtained on finite-dimensional lattices, although the initial formation of small clusters of agents reaching a local consensus through repeated interactions is similar. While a majority of nodes soon compose small clusters, the fraction of nodes with more words is not negligible and decreases only at the end of the evolution. Therefore, the dynamics cannot be seen as a coarsening or growth of clusters but as a slow process of correlations between inventories, in a way much more similar to what is observed in mean field [13].

C. Effect of the degree heterogeneity

In regular topologies, as well as in mean field, all agents face an identical environment. Complex networks are different in that respect, and strong differences in behavior can be expected for agents sitting on nodes with large or small degrees. Global properties of dynamical processes are often affected by the heterogeneous character of the network topology [16,17]. The previous subsection, has shown that, similarly to what happens for the Voter model [35], the dynamics of the naming game is similar on heterogeneous and homogeneous networks. Nonetheless, a more detailed analysis reveals that agents with different degrees present very different activity patterns, whose characterization is necessary to get additional insights on the naming game dynamics [37].

Let us first consider the average success rate $S_k(t)$ of nodes of degree k. At the early stages of the dynamics it can be computed following the arguments of Sec. V A. The probability of choosing twice the edge i-j is

$$\frac{t}{N}\left(\frac{1}{k_i} + \frac{1}{k_j}\right), \tag{1}$$

i.e., the probability of choosing first i $(1/N)$ then j $(1/k_i)$ or vice versa. Neglecting the correlations between k_i and k_j, one
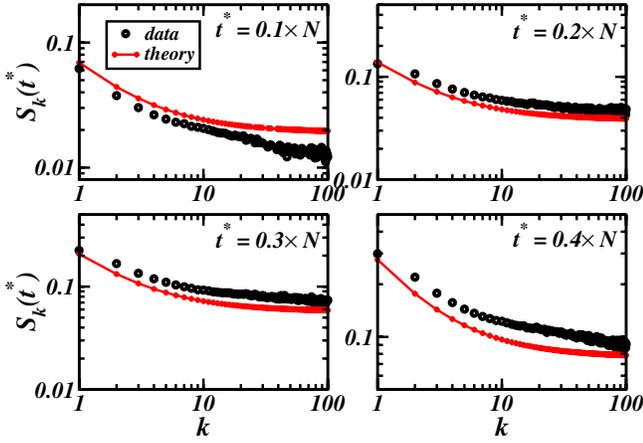
FIG. 9. (Color online) The four plots correspond to snapshots of the success rate per degree classes $S_k(t^*)$ at times $t^*/N=0.1$, 0.2, 0.3, and 0.4 in a power-law random graph with exponent $\gamma=2.2$ and size $N=10^4$ generated using the uncorrelated configuration model. The numerical computation of the success rate (black circles) qualitatively agrees with the numerical evaluation of the expression in Eq. (2) (red crosses).

can average over all nodes i of fixed $k_i=k$, obtaining

$$S_k(t) \simeq \frac{t}{N}\left(\frac{1}{k} + \left\langle\frac{1}{k}\right\rangle\right). \tag{2}$$

Figure 9 shows that, on uncorrelated scale-free networks (UC model), the data (circles) obtained by numerical simulations are in qualitative agreement with the direct calculations of the expression in Eq. (2) (crosses). These data together with Eq. (2) show that, at the very beginning, the success rate grows linearly but the effect of the degree heterogeneity is partially screened by the presence of the constant term $\langle 1/k\rangle$. The same argument can be used to predict that the success rate should be essentially degree independent for larger times. $S(t)$ is indeed always given by two terms, of which only that referring to the node playing as speaker contains an explicit dependence on $1/k$. The argument is only approximate since the multiplicative prefactors contain non-negligible correlations due to the overlapping inventories. More precisely, these arguments are correct for a neutral naming game rule, but they should hold also for the direct naming game in which the constant term, coming from the activity of nodes as hearers, is much more relevant for high degree nodes.

Another interesting point concerns the height of the memory peak. Looking at classes of nodes of a given degree, we get that the height of the memory peak is larger for nodes of a larger degree, as shown in Fig. 10. This can be understood by the fact that hubs act more frequently as hearers and therefore receive and collect the different words created in the various "area" of the network they connect together [44]. In fact, the maximal memory used by a node of degree k is proportional to $\sqrt{k}$ (see bottom panel in Fig. 10). For the mean-field case, all agents have degree $k=N-1$ and the maximal value of the total memory N_w scales indeed as $N\sqrt{k}=N^{3/2}$. Note, however, that in the general case, the esti-
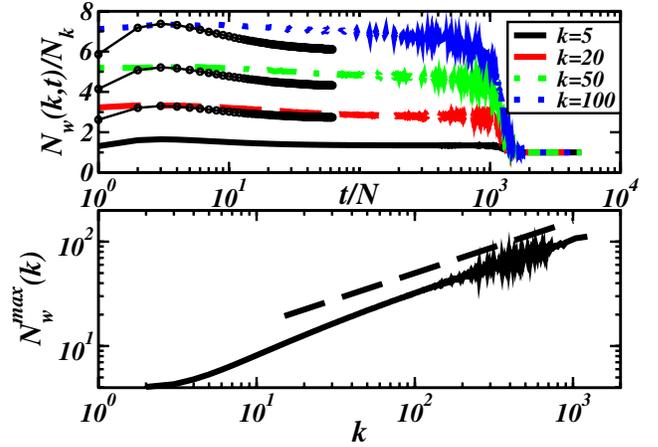
FIG. 10. (Color online) BA model with $m=2$ (i.e., $\langle k\rangle=4$), $N=5\times10^4$. Bottom: maximum memory used by a node as a function of its degree. The dashed line is $\propto\sqrt{k}$. Top: average memory used by nodes of degree k, for various values of k. The lines show the total memory $N_w(k,t)$ used by nodes of degree k at time t, normalized by the number N_k of nodes of degree k. The circles correspond to the bottom curve ($k_0=5$) rescaled by $\sqrt{k/k_0}$ showing the scaling of the peaks. Note that the values of $N_w(k,t)/N_k$ are averages over many runs that wash out fluctuations and therefore correspond to smaller values than the extremal values observed for $N_w^{max}(k)$.

mation of the peak of N_w is not as straightforward. This peak is indeed a convolution of the peaks of the inventory sizes of single agents, that have distinct activity patterns and may reach their maximum in memory at different temporal steps.

The knowledge of the average maximal memory of a node of degree k is not sufficient to understand which degree classes play a major role in driving the dynamics towards the consensus. More insights on this issue can be obtained observing the behavior of the total number of different words in each degree class. Figure 11 shows the evolution of the number $N_d(k,t)$ of different words in the class of nodes with degree k, for various values of k in a BA network with size
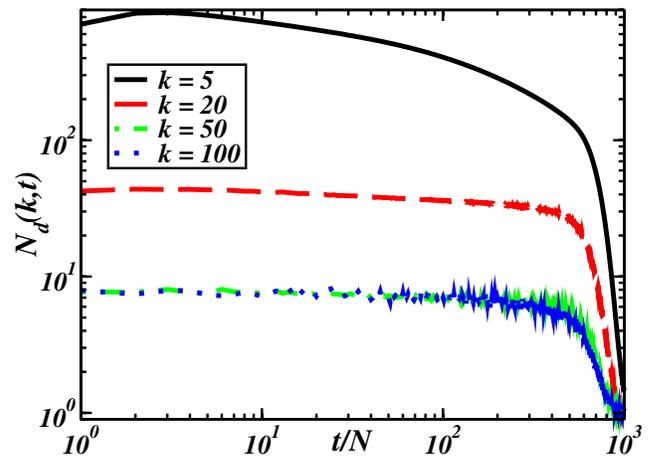
FIG. 11. (Color online) BA network, $m=2$, $N=10^4$. Total number of different words in classes of degree k, for values of $k=5$, 20, 50, and 100 (curves for $k=50$ and 100 are almost on top of each other).

$N=10^4$ and $\langle k \rangle = 4$. Two competing effects take part in determining the differences between nodes: high-degree nodes require more memory than low-degree nodes (Fig. 10), but their number is much smaller. As a result, low-degree classes have, in fact, overall a larger number of different words (as shown in Fig. 11). This is due to the fact that during the initial phase, in which words are invented, low-degree nodes are more often chosen as speakers and invent many different words. The hubs need each a larger memory but they, in fact, retain a smaller number of different words. After the peak in memory, the dynamical evolution displays a relatively fast decrease of $N_d(k,t)$ for small k while a plateau is observed at large k: words are progressively eliminated for low-k nodes while the hubs, which act as intermediaries and are in contact with many agents, still have typically many words in their inventories. The role of the hubs, then, is that of diffusing words throughout the network and their property of connecting nodes with originally different words helps the system to converge. On the other hand, however, playing mostly as hearers, the hubs are not able to promote actively successful words, and their convergence follows that of the neighboring low-degree sites. In fact, once the low-degree nodes have successfully eliminated most of the different words created initially, the system globally converges on a faster time scale. We note that the average memory $N_w(k,t)/N_k$ converges slightly faster than $N_d(k,t)$ (and that $N_d(k,t)$ converges faster for larger k), showing that the very final phase consists in the late adoption of the consensus by the lowest degree nodes, in a sort of final cascade from the large to the small degrees.

D. Effect of the average degree and clustering

Social networks are generally sparse graphs, but their structure is often characterized by high local cohesiveness, that is the result of a very natural transitive property of many social interactions [18]. The simplest way to take into account these features on the dynamics of naming game is that of studying the effects of changing the average degree and the clustering coefficient of the network.

The effects of increasing the average degree on the behavior of the main global quantities are reported in Fig. 12. In both ER (left) and BA (right) models, increasing the average degree provokes an increase in the memory used, while the global convergence time is decreased. Note also that, while the behavior of the convergence time with N (i.e., a power-law N^β with $\beta \approx 1.4$) is very robust, the linear scaling for the memory peak properties ($N_w^{max} \propto N^\alpha$ and $t_{max} \propto N^\alpha$ with $\alpha = 1$), are slightly altered by an increase in the average degree (not shown). Increasing $\langle k \rangle$ at finite N brings indeed the system closer to the mean-field behavior where the scaling of these quantities is nonlinear ($\alpha_{MF} = 1.5$); for fixed $\langle k \rangle$ however, the linear scaling is recovered at large enough sizes.

Moreover, for larger average degree, the number of nodes having only one word decreases (not shown); i.e., the system needs a more complicated reorganization phase that involves a larger number of agents with many words, but induces a faster convergence. In fact, the larger possibilities of interaction given by the larger number of connections allows for a better sharing of common words and for a more efficient
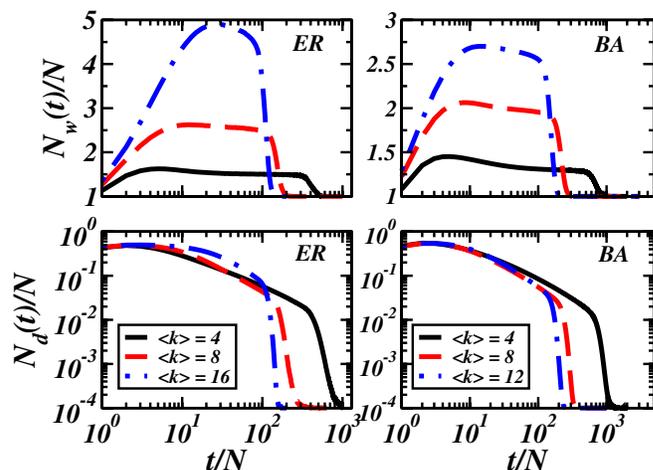
FIG. 12. (Color online) ER networks (left) and BA networks (right) with $N=10^4$ agents and average degree $\langle k \rangle = 4$, 8, and 16. The increase of average degree leads to a larger memory used (N_w, top) but a faster convergence. The maximum in the number of different words is not affected by the change in the average degree (bottom).

correlation of inventories, thus favoring a faster convergence.

Note that the clustering is slightly changing when changing the average degree, but its variation is small enough for the two effects to be studied separately. Here we use some other mechanisms to enhance clustering, summarized in the following two models that have been defined in Sec. IV: CER random graphs, and mixed BA-DMS model.

Figure 13 shows the effect of increasing the clustering at fixed average degree and degree distributions: the number of different words is not changed, but the average memory used is smaller and the convergence takes more time. Moreover, the memory peak at fixed k is smaller for larger clustering (not shown): it is more probable for a node to speak to two neighbors that share common words because they are themselves connected and have already interacted, so that it is less probable to learn new words. Favored by the larger number of triangles, cliques of neighboring nodes learn from the start the same word, causing a slight increase in the fraction of nodes with only one word as reported in Fig. 14 for both homogeneous and heterogeneous networks. At a fixed average degree, i.e., global number of links, less connections are moreover available to transmit words from one part of the network to the other since many links are used in "local" triangles. The enhanced local coherence, therefore, is in the long run an obstacle to the global convergence. We note that this effect is similar to the observation by Holme *et al.* [36] that, at fixed $\langle k \rangle$, more clustered networks are more vulnerable to attacks since many links are "wasted" in redundant local connections.

E. Effect of hierarchical structures

In the previous sections we have argued that networks with small-world property have fast (mean-field-like) convergence after a reorganization phase whose duration de-
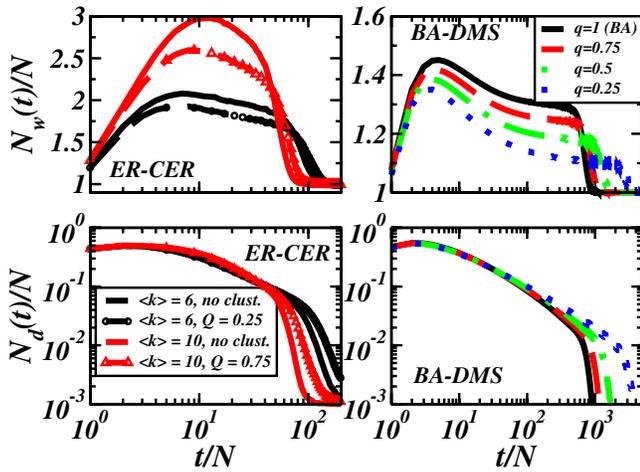
FIG. 13. (Color online) Effect of clustering on the behavior of the total number of words $N_w(t)$ and of the number of different words $N_d(t)$ on random graphs (left) and scale-free networks (right) with $N=10^4$. In order to inject triangles into the ER random graphs we have used the construction described in Sec. IV, obtaining clustered random graphs (CER model, with clustering coefficient proportional to Q) that have been compared to standard ER graphs with equal average degree ($\langle k \rangle = 6$ and 10). Scale-free networks have been generated with the mixed BA-DMS model described in Sec. IV, in which the clustering coefficient is proportional to $1-q$. In both networks higher clustering leads to smaller memory capacity required but a larger convergence time.

FIG. 14. (Color online) Effect of enhanced clustering on the fraction of agents with 1 (n_1 in black), 2 (n_2 in red), and 3 (n_3 in blue) words. Top: We compare a clustered random graph (CER model, with clustering coefficient proportional to Q) to standard ER graphs, both with average degree $\langle k \rangle = 10$. Bottom: Scale-free networks have been generated with the mixed BA-DMS model described in Sec. IV, in which the clustering coefficient is proportional to $1-q$. In both cases there is a tendency to increase the fraction of agents with one word and decrease the other fractions.

pends on other properties of the system. The small-world property holds when the diameter of the network grows slowly, i.e., logarithmically or slower, with its size N. This ensures that every part of the network is rapidly reachable from any other part, in contrast to what happens with regular lattices. Such property, therefore, generically enhances the possibility of creating correlations between the inventories of the agents and of finally converging to a consensus. In this subsection, we show that this line of reasoning bears, surprisingly at first sight, some exceptions.

The first (and easiest to apprehend) exception is given by the scale-free trees, obtained by the preferential attachment procedure with $m=1$. In this case, as shown in Fig. 15, the convergence is reached very slowly, with $N_w(t)/N-1$ decreasing as a power law of the time. This is in contrast with the generic behavior, i.e., a plateau followed by an exponential convergence, as shown also for reference in Fig. 15, but similar to the finite-dimensional lattices (the average cluster size as well grows as a power-law, in contrast with the data of Fig. 7). This is reminiscent of what happens for the Voter model [35], in which a power law instead of exponential decrease of the fraction of active bonds is observed, and can be understood through the tree structure of the network. Indeed, from the viewpoint of the dynamics, a tree is formed by two ingredients: linear structures on which the interfaces between clusters diffuse as in one-dimensional systems and branching points at which interfaces may be pinned and their motion slowed. In fact, we have checked that similar (slow) power-law behaviors are also obtained for the naming game on a Cayley tree (i.e., in which every node has the same

degree) or for scale-free trees with different degree distributions (obtained through the generalized linear preferential attachment model).

The slowness of this dynamical behavior is, however, rooted in a slightly more subtle consideration. As Fig. 15 indeed shows, the naming game displays power-law convergence in other heterogeneous networks that are not at all tree-like, such as the DMS model with $m=2$ [28], in which at
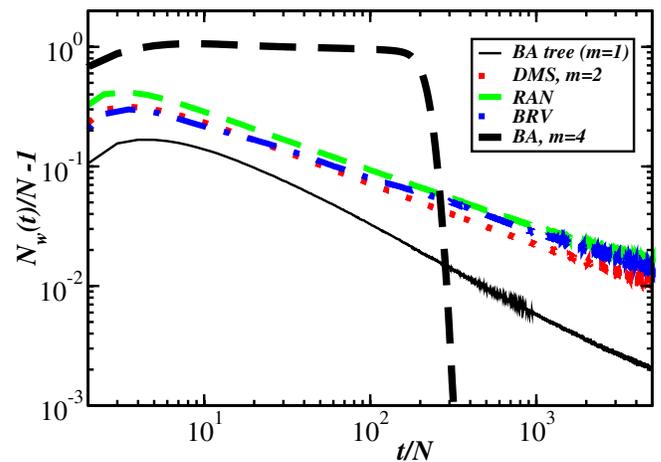
FIG. 15. (Color online) Evolution of the number of words for the naming game on hierarchical networks, namely the BA tree, the DMS model, the hierarchical BRV [37], and the random Apollonian network. The case of the BA model with $m=4$ is shown with a dashed line for reference.

each step a triangle is created, the deterministic scale-free networks of Barabási, Ravasz, and Vicsek (BRV) [38], or the Apollonian and random Apollonian networks (RAN) [39,40]. Let us briefly recall how these networks are constructed.

(1) For the DMS model with $m=2$, one adds at each step a new node which is connected to the extremities of a randomly chosen edge.

(2) For the deterministic scale-free BRV networks, one starts (step 1) with two nodes connected to a root. At each step n, two units (of 3^{n-1} nodes) identical to the network formed at the previous step are added, and each of the bottom 2^n nodes are connected to the root.

(3) The random Apollonian networks are embedded in a two-dimensional plane. One starts with a triangle; a node is added in the middle of the triangle and connected to the three previous nodes; at each step, a new node is added in one of the existing triangles (chosen at random) and connected to its three corners, replacing the chosen triangle by three new smaller triangles.

All these networks share a very important and hard to quantify property: they are hierarchically built. This is particularly clear for the BRV case, since at each step the new network is formed by three identical sub networks. In the RAN as well, hierarchically nested units can be identified with the triangles, each of which contains other smaller triangles. Finally, in the DMS case, one can identify a unit at a certain scale as an edge and the set of nodes that have been attached to the extremities of this edge or of the edges subsequently created in this unit. Because of these particular network organizations, each node belongs, in fact, to a given sub-hierarchical unit and, to go from one node to another node in another subunit, a hierarchical path has to be followed. The trees represent a particular class of such structures, in which there exist only one path between two given nodes. In this sense, such networks, although being small-world, present a structure which renders communication between different parts of the network more difficult. Each sub-unit can, therefore, converge towards a local consensus which renders the global consensus more cumbersome to achieve.

Such results show that the small-world property, in fact, does not by itself guarantee an efficient convergence of dynamical processes such as the naming game, and that strongly hierarchical structures, in fact, slow down and obstruct such convergence.

VI. DISCUSSION AND OUTLOOK

In this paper, we have studied the dynamical properties of the minimal naming game model [13] in populations of agents interacting on complex topologies, focusing on homogeneous and heterogeneous networks (represented, respectively, by the Erdös-Rényi random graph model and by the Barabási-Albert scale-free model). Social networks indeed are typically neither fully connected graphs nor regular lattices. We have considered the effects of various network characteristics such as the heterogeneity, the average degree, and the presence of clustering.

The main characteristic of the studied networks is the small-world property (the average hopping distance between two nodes scales only logarithmically with the size of the network). After an initial phase during which words are created, the small-world property ensures their propagation out of the local scale, boosting up the spreading process contrarily to what happens in low-dimensional lattices where words' spreading is purely diffusive (see Secs. V A and V B). As already suggested in Ref. [34], we argue that the small-world property allows inhomogeneous and sparse networks to recover the high temporal efficiency observed in the mean-field system. For both the ER and BA network models we get a scaling law for the convergence time t_{conv} with the size N of the system of the type $t_{conv} \sim N^{\beta_{SW}}$, with the exponent approximately $\beta_{SW} \simeq 1.4$. The discrepancy with the mean-field exponent ($\beta_{MF} \simeq 1.5$) may be due to logarithmic corrections that are unlikely to be captured using numerical scaling techniques. Moreover, small-world networks have higher memory efficiency than the mean-field model, since the peak in the total number of words scales only linearly with the size N. This is due to the fact that these networks are *sparse*, i.e., their average degree $\langle k \rangle$ is small compared to N.

The detailed analysis of the naming game dynamics shows distinct activity patterns on homogeneous and heterogeneous networks. In homogeneous networks all nodes have a similar neighborhood and therefore similar dynamical evolution, while in heterogeneous networks classes of nodes with different degrees play different roles in the evolution of the game. The role of the hubs is better understood thanks to the degree based analysis of the number of words and different words. High-degree nodes, indeed, are more likely chosen as hearers and, consequently, they have larger inventory sizes. At the beginning, because of the pair choosing strategy (direct naming game), low-degree nodes are much more involved in the process of word generation than the hubs. Local consensus is easily reached and a large amount of locally stable different words get in touch with higher-degree nodes. The latter start to accumulate a large number of words in their inventories, playing as spreaders of names towards less connected agents and finally driving the convergence. From this viewpoint, the convergence dynamical pattern of the naming game on heterogeneous complex networks presents some similarities with more studied epidemic spreading phenomena [41]. A more detailed comparison of the activity pattern for the direct and reverse naming game is left for future work [37].

The relation between topological properties and the dynamical evolution of the system are further characterized by a detailed study of the effects of varying the average degree and clustering coefficient. These effects are equivalent on homogeneous and heterogeneous networks. While any increase of the average degree provokes a larger memory peak and a faster convergence, the growth of clustering coefficient leads to the decrease of the necessary memory but the fast obtention of local consensus delays in the long run the global convergence. The latter effect is particularly relevant for real social networks in which local cohesiveness is an important feature that cannot be neglected. Another important ingredient of real networks that we have not addressed here is the presence of degree correlations in the network topology. It would indeed be interesting to know in what measure positive or negative degree correlations affect the negotiating processes of the agents.

In summary, as other models of opinion formation [12], the naming game shows a nonequilibrium dynamical evolution from a disordered state to a state of global agreement. However, with respect to most opinion models, in which the agents may accept or refuse to conform to the opinion of someone else, the naming game gives more importance to the bilateral negotiation process between pairs of agents that is a cornerstone in the evolutionist theory of language [1]. For this reason the naming game should be regarded as an independent attempt to model the ultimate emergence of a globally accepted linguistic convention or, in other terms, the establishment of a self-organized communication system. In contrast with other nonequilibrium models, as those based on zero-temperature Glauber dynamics or the voter model [35,42], we do not find any signature of the occurrence of metastable blocked states in any relevant topology with quenched disorder. While the total number of words displays a plateau whose length increases with the system size during the reorganization phase, indeed, the number of different words is continuously decreasing, revealing that the convergence is not a matter of fluctuations due to finite-size effects, but the result of an evolving self-organizing process. Such behavior makes the naming game a robust model of self-coordinated communication in any structured population of agents. A noticeable exception concerns the case of agents sitting on networks with strong community structures, i.e., networks composed of a certain number of internally highly connected groups interconnected by few links working as bridges. Figure 16 reports the behavior of the naming game on such a network, composed of fully connected cliques, each of c nodes, the various cliques being connected to each other with only one link. From simulations it turns out that, not only the total number of words, but even the number of different words display a plateau whose duration increases with the size of the system. The number of different words in the plateau equals the number of communities, while the corresponding total number of words per node is about one, proving the existence of a real metastable state in which the system reaches a long-lasting multi-vocabulary configuration. Indeed, each community reaches internal consensus but the weak connections between communities are not sufficient for words to propagate from one community to the other.
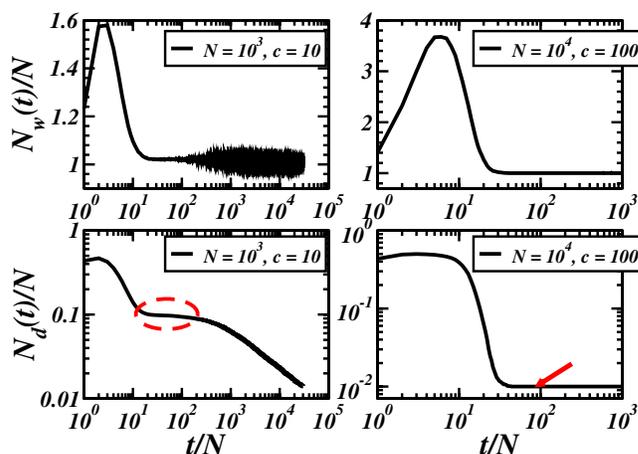
FIG. 16. (Color online) Metastable states in networks with strong community structure. Each community is composed of c nodes so that there are $N/c = 100$ communities. The dashed ellipse and the arrow identify the metastable states with $N_d = N/c$ different words.

In conclusion, populations of agents with fixed complex topology do evolve towards a homogeneous state of consensus and efficient communication, except for somehow artificial network structures, the detailed topological properties affecting only the convergence pattern and time scale. Future work will address the important issue of a possible interplay between topology and dynamics in populations in which the agents are free of rearranging their connectivity patterns in relation to local (or global) information on the dynamical evolution of the system. It would for example be interesting to verify if such interplay may allow for a natural emergence of community structures and multi-language cohabitation.

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[43] Note that, in order to compare an ER and a CER network with the same $\langle k \rangle$, we therefore tune p for the construction of the corresponding CER.

[44] Note that for the reverse Naming Game, the hubs act more frequently as speakers and therefore accumulate much less different words. The required maximal memory then slightly decreases at large k.